

NONCONVEX LIPSCHITZ FUNCTION IN PLANE WHICH IS LOCALLY CONVEX OUTSIDE A DISCONTINUUM

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ABSTRACT. We construct a Lipschitz function on \mathbb{R}^2 which is locally convex on the complement of some totally disconnected compact set but not convex. Existence of such function disproves a theorem that appeared in a paper by L. Pasqualini and was also cited by other authors.

1. INTRODUCTION

In his work from 1938 L. Pasqualini presents a theorem (see [3, Theorem 51, p. 43]) of which the following statement is a reformulation:

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function and $M \subset \mathbb{R}^d$ a set not containing any continuum of topological dimension $(d - 1)$. Suppose that f is locally convex on the complement of M . Then f is convex on \mathbb{R}^d .

The proof however contains a gap. This result also appeared in the survey paper [1], where the (incorrect) proof was shortly repeated. Also V.G. Dmitriev mentions this result in [2], although he provides a wrong reference.

As a counterexample to the theorem of Pasqualini we present the following theorem:

Theorem 1.1. *There is a Lipschitz function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $M \subset \mathbb{R}^2$ such that*

- *f is locally convex on $\mathbb{R}^2 \setminus M$,*
- *f is not convex on \mathbb{R}^2 ,*
- *M is compact and totally disconnected,*
- *f has compact support.*

Note that it is simple observation that such set M cannot be of one dimensional Hausdorff measure 0 (this fact actually essentially follows from the original argument by Pasqualini).

In this situation it seems natural to call a compact set M *convex nonremovable* if there is a nonconvex say Lipschitz function f which is locally convex on the complement of M . Note that in such context it may be relevant that the function from Theorem 1.1 is Lipschitz (or continuous) or that it has a compact support or that it is defined on whole \mathbb{R}^2 , since it is possible that such notion of nonremovability might differ if we a priori assume some of those conditions to hold for f . In some sense the set M from Theorem 1.1 may be considered as nonremovable in one of the strongest ways possible.

2. PRELIMINARIES

In the paper we will use the following more or less standard notation and definitions:

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For $a, b \in \mathbb{R}^d$ and $r > 0$ we will denote by $B(a, r)$ the closed ball with center a and radius r and $[a, b]$ will denote the closed line segment with endpoints a and b . For $A \subset \mathbb{R}^d$ the symbol $\text{co}A$ will mean the convex hull of A and A^c will mean the complement of A . If $l \subset \mathbb{R}^2$ is a line and $\varepsilon > 0$ then we define $l(\varepsilon) = \{x \in \mathbb{R}^2 : \text{dist}(x, l) < \varepsilon\}$.

A function f defined on a set $A \subset \mathbb{R}^2$ is called L -Lipschitz, if for every $x, y \in A$, $x \neq y$, we have $|f(x) - f(y)| \leq L|x - y|$.

We will call f locally convex on A if for every x, y such that $[x, y] \subset A$ and $\alpha \in [0, 1]$ we have $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$.

Finally, f will be called piecewise affine on A if there is a locally finite triangulation Δ of A such that f is affine on every triangle from Δ .

3. CONSTRUCTION OF THE FUNCTION

Definition 3.1. Let \mathcal{Q} be a system of all unions of finite systems of (closed) polytopes in \mathbb{R}^2 . Let $L > 0$, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $P \in \mathcal{Q}$. We say that a pair (P, f) is L -good if

- (1) f is L -Lipschitz,
- (2) f is piecewise affine on P^c ,
- (3) f is locally convex on P^c .

The key technical result is the following:

Lemma 3.2. Let $\varepsilon, L > 0$, l line in \mathbb{R}^2 let (P, g) be a L -good. Then there is an $(L + \varepsilon)$ -good pair (Q, h) such that

- (1) $Q \subset P$,
- (2) $h = g$ on P^c ,
- (3) if $x, y \in Q$ belong to the different component of $\mathbb{R}^2 \setminus l(\varepsilon)$ then they belong to the different component of Q .

We first prove Theorem 1.1 using Lemma 3.2

Proof of Theorem 1.1. Choose a sequence $\{x_n\}_{n=1}^\infty$ dense in the plane and consider any sequence of lines $\{l_n\}_{n=1}^\infty$ with the property that for any $i, j \in \mathbb{N}$ there is some $k \in \mathbb{N}$ such that $x_i, x_j \in l_k$. Choose a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0, \infty)$ such that $\sum_{n=1}^\infty \varepsilon_n < \infty$. Then the sequence $\{l_n(\varepsilon_n)\}_{n=1}^\infty$ has the property that for every $x, y \in \mathbb{R}^2$, $x \neq y$, there is some $k \in \mathbb{N}$ such that x and y belong to the different component of $\mathbb{R}^2 \setminus l_k(\varepsilon_k)$.

We will proceed by induction and construct a sequence of functions $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a sequence $P_i \subset \mathcal{Q}$, $i = 0, 1, \dots$, such that for every i the following conditions hold:

- (1) pair (P_i, f_i) is $(1 + \sum_{n=1}^i \varepsilon_n)$ -good,
- (2) if $i > 0$ then $P_i \subset P_{i-1}$,
- (3) if $i > 0$ then $f_i = f_{i-1}$ on $(P_{i-1})^c$,
- (4) if $i > 0$ and if $x, y \in P_i$ belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$ then they belong to the different component of P_i .

To do this let f_0 be an arbitrary 1-Lipschitz function on \mathbb{R}^2 which is equal to 0 on $((-3, 3)^2)^c$ and equal to 1 on $[-1, 1]^2$ and put $P_0 := [-3, 3]^2 \setminus (-1, 1)^2$. Validity of conditions (1) – (4) is obvious.

Now, if we have f_{i-1} and P_{i-1} constructed we obtain f_i and P_i simply by applying lemma 3.2 with $\varepsilon = \varepsilon_i$, $L = (1 + \sum_{n=1}^{i-1} \varepsilon_n)$, $l = l_i$, $P = P_{i-1}$ and $g = f_{i-1}$. The function f_i will be then equal to h from the statement of lemma 3.2 and P_i will be equal to the corresponding Q . Validity of conditions (1) – (4) follows directly from lemma 3.2.

Put $M := \cap P_i$. Due to property (2) M is compact and nonempty. To prove that M is totally disconnected consider $x, y \in M$, $x \neq y$. By the choice of the sequences $\{l_n\}_{n=1}^\infty$ and $\{\varepsilon_n\}_{n=1}^\infty \subset \mathbb{R}^+$ there is some i such that x and y belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$. By property (3) we have that x and y belong to the different component of P_i . Using property (2) again we then obtain that x and y belong to the different component of M as well.

Define $\tilde{f} : M^c \rightarrow \mathbb{R}$ in such a way that $\tilde{f}(x) = f_i(x)$ whenever $x \in (P_i)^c$. It is easy to see that the definition of \tilde{f} is correct due to properties (2) and (3) and the definition of M , and also that by property (1) the function \tilde{f} is $(1 + \sum_{n=1}^\infty \varepsilon_n)$ -Lipschitz and locally convex on M^c . By Kirszbraun's theorem there is a $(1 + \sum_{n=1}^\infty \varepsilon_n)$ -Lipschitz function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f = \tilde{f}$ on M^c . Therefore f is locally convex on M^c as well. Also, f has compact support due to properties (2) and (3), the fact that P_0 is compact and that f_0 is supported in P_0 .

It remains to show that f is not convex on \mathbb{R}^2 , but this is easy since

$$\frac{f(-3, 0) + f(3, 0)}{2} = 0 < 1 = f(0, 0).$$

□

The proof of Lemma 3.2 is divided into several lemmatae.

Lemma 3.3. *Let $H \subset \mathbb{R}^2$ be a closed halfplane, $x \in \mathbb{R}^2 \setminus H$ and $L > 0$. If $f : H \cup \{x\} \rightarrow \mathbb{R}$ is L -Lipschitz and affine on H , then for every $y \in \partial H$ the function*

$$g_y(z) = \begin{cases} f(z), & \text{if } z \in H, \\ \alpha f(x) + (1 - \alpha)f(y), & \text{for } z = \alpha x + (1 - \alpha)y, \alpha \in [0, 1]. \end{cases}$$

is L -Lipschitz as well.

Proof. Without any loss of generality we can suppose that $H = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$, $f(y) = 0$ and that $y = (0, 0)$. This means that g_y is in fact linear on both H and $[x, y]$. Choose $a \in H$ and $b = \alpha x$ for some $\alpha \in [0, 1]$. Now,

$$\begin{aligned} |g_y(a) - g_y(b)| &= \alpha \left| g_y\left(\frac{1}{\alpha}a\right) - g_y\left(\frac{1}{\alpha}b\right) \right| = \alpha \left| g_y\left(\frac{1}{\alpha}a\right) - g_y\left(\frac{1}{\alpha}\alpha x\right) \right| \\ &= \alpha \left| g_y\left(\frac{1}{\alpha}a\right) - g_y(x) \right| \leq \alpha L \left| \frac{1}{\alpha}a - x \right| = \alpha L \left| \frac{1}{\alpha}a - \frac{1}{\alpha}\alpha x \right| \\ &= L|a - \alpha x| = L|a - b|. \end{aligned}$$

Similarly, if $a = \alpha x$ and $b = \beta x$ for some $\alpha, \beta \in [0, 1]$ $\alpha \neq \beta$ we have

$$|g_y(a) - g_y(b)| = |\alpha f(x) - \beta f(x)| \leq |\alpha - \beta|f(x) \leq |\alpha - \beta|L.$$

□

Lemma 3.4. *Let $\varepsilon, L, K > 0$. Let f be a L -Lipschitz function on $[-K, K]^2$, which is equal to an affine function f_1 on $[-K, 0] \times [-K, K]$, and $z \in (0, K) \times (-K, K)$. Then there is an $x \in [(0, 0), z]$ and $\gamma > 0$ such that for every $y \in B(x, \gamma)$ and every $w \in B((0, 0), \gamma) \cap (\{0\} \times (-K, K))$ the function*

$$g_{y,w}(u) = \begin{cases} f(u), & \text{if } u \in [-K, 0] \times [-K, K], \\ \alpha f(w) + (1 - \alpha)f(y), & \text{for } u = \alpha w + (1 - \alpha)y, \alpha \in [0, 1]. \end{cases}$$

is $(L + \varepsilon)$ -Lipschitz and $|g_{y,w} - f| < \varepsilon$ on $[-K, 0] \times [-K, K] \cup [w, y]$.

Proof. Without any loss of generality we can suppose that $K = L = 1$ and that $f(0, 0) = 0$. Since f is 1-Lipschitz we can find a sequence $\{x_i\}_{i=1}^\infty \subset [(0, 0), z]$ converging to $(0, 0)$ such that for some $s \in [-1, 1]$

$$(3.1) \quad s_i := \frac{f(x_i)}{|x_i|} \rightarrow s \quad \text{as } i \rightarrow \infty.$$

Consider now the sequence of functions $h_i : [-\frac{1}{|x_i|}, 0] \times [-\frac{1}{|x_i|}, \frac{1}{|x_i|}] \cup \{\frac{z}{|z|} =: \tilde{z}\} \rightarrow \mathbb{R}$ defined as

$$h_i(u) := \frac{1}{|x_i|} f(|x_i|u).$$

Then h_i is 1-Lipschitz for every i . Since f is equal to an affine function f_1 on $[-1, 0] \times [-1, 1]$ and $f(0, 0) = 0$ we have $h_i = f_1$ on $[-\frac{1}{|x_i|}, 0] \times [-\frac{1}{|x_i|}, \frac{1}{|x_i|}]$. Also $h_i(\tilde{z}) = s_i$. Therefore by (3.1) the function $h : (-\infty, 0] \times (-\infty, \infty) \cup \{\tilde{z}\} \rightarrow \mathbb{R}$ which is equal to f_1 on $(-\infty, 0] \times (-\infty, \infty)$ and such that $h(\tilde{z}) = s$ is also 1-Lipschitz.

Consider $\tilde{\gamma} > 0$ such that $\tilde{\gamma} < \frac{\varepsilon \tilde{z}_1}{4}$ (here by \tilde{z}_1 we mean the first coordinate of \tilde{z}) and such that $\frac{|v - \tilde{z}|}{|v - \tilde{z}| - \tilde{\gamma}} < 1 + \frac{\varepsilon}{2}$ for every $v \in (-\infty, 0] \times (-\infty, \infty)$.

Now, for every $\tilde{s} \in [s - \tilde{\gamma}, s + \tilde{\gamma}]$, $v \in (-\infty, 0] \times (-\infty, \infty)$ and $u \in B(\tilde{z}, \tilde{\gamma})$

$$\begin{aligned} \frac{f_1(v) - \tilde{s}}{|v - u|} &\leq \frac{|f_1(v) - s|}{|v - u|} + \frac{|s - \tilde{s}|}{|v - u|} \leq \frac{|f_1(v) - s|}{|v - \tilde{z}| - \tilde{\gamma}} + \frac{\tilde{\gamma}}{|v - \tilde{z}| - \tilde{\gamma}} \\ &\leq \frac{|f_1(v) - s|}{|v - \tilde{z}|} \cdot \frac{|v - \tilde{z}|}{|v - \tilde{z}| - \tilde{\gamma}} + \frac{2\tilde{\gamma}}{\tilde{z}_1} \leq \left(1 + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = 1 + \varepsilon. \end{aligned}$$

Therefore, by lemma 3.3 for every $\tilde{s} \in [s - \tilde{\gamma}, s + \tilde{\gamma}]$, $v \in \{0\} \times (-\infty, \infty)$ and $t \in B(\tilde{z}, \tilde{\gamma})$ the function

$$\tilde{h}_{v,t,\tilde{s}}(u) = \begin{cases} f_1(u), & \text{if } u \in (-\infty, 0] \times (-\infty, \infty), \\ (1 - \alpha)\tilde{s} + \alpha f_1(v), & \text{for } u = (1 - \alpha)t + \alpha v, \alpha \in [0, 1]. \end{cases}$$

is $(1 + \varepsilon)$ -Lipschitz as well.

Choose i such that $s_i \in [s - \frac{\tilde{\gamma}}{2}, s + \frac{\tilde{\gamma}}{2}]$ and put $x = x_i$ and $\gamma = \frac{|x|\tilde{\gamma}}{2}$. Now, consider some $y \in B(x, \gamma)$ and some $w \in B((0, 0), \gamma) \cap \{0\} \times (-1, 1)$ and let $g_{y,w}$ be as in the statement on the lemma. First we will prove that $g_{y,w}$ is $(1 + \varepsilon)$ -Lipschitz. To do this we first observe that $\frac{1}{|x|}g_{y,w}(\frac{\cdot}{|x|})$ is equal to $\tilde{h}_{\frac{w}{|x|}, \frac{y}{|x|}, \frac{f(y)}{|x|}}(\cdot)$, where the first function is defined. Now, we have $\frac{w}{|x|} \in \{0\} \times (-\infty, \infty)$,

$$\left| \frac{y}{|x|} - \tilde{z} \right| = \left| \frac{y}{|x|} - \frac{x}{|x|} \right| = \frac{|y - x|}{|x|} \leq \frac{|x|\tilde{\gamma}}{2|x|} \leq \tilde{\gamma},$$

which means $\frac{y}{|x|} \in B(\tilde{z}, \tilde{\gamma})$ and finally

$$\begin{aligned} \left| \frac{f(y)}{|x|} - s \right| &= \left| \frac{f(y) - f(x) + f(x)}{|x|} - s \right| \leq \left| \frac{f(y) - f(x)}{|x|} \right| + \left| \frac{f(x)}{|x|} - s \right| \\ &\leq \frac{|y - x|}{|x|} + \frac{\tilde{\gamma}}{2} \leq \frac{\frac{|x|\tilde{\gamma}}{2}}{|x|} + \frac{\tilde{\gamma}}{2} = \frac{\tilde{\gamma}}{2} + \frac{\tilde{\gamma}}{2} = \tilde{\gamma}. \end{aligned}$$

which means that $\frac{f(y)}{|x|} \in [s - \tilde{\gamma}, s + \tilde{\gamma}]$ and we are done since $\frac{1}{|x|}g_{y,w}(\frac{\cdot}{|x|})$ and $g_{y,w}$ have the same Lipschitz constant.

To finish the proof it is now sufficient to observe that if we additionally choose x_i small enough we obtain also $|g_\varepsilon - f| < \varepsilon$ on $[-1, 0] \times [-1, 1] \cup [w, y]$. \square

Lemma 3.5. *Let $L, \varepsilon, \delta > 0$, $a < b$ and $c < d$ be given. Let*

$$P = \text{co}\{(-1, a), (-1, b), (1, c), (1, d)\}$$

and

$$P^\varepsilon = \text{co}\{(-1, a - \varepsilon), (-1, b + \varepsilon), (1, c - \varepsilon), (1, d + \varepsilon)\}.$$

Suppose that f is a L -Lipschitz function defined on \mathbb{R}^2 which is locally affine on $P^\varepsilon \setminus P$. Then there are

$$\frac{a+c}{2} =: a_0 < a_1 < \dots < a_{n-1} < a_n := \frac{b+d}{2}$$

and $\frac{1}{2} > \kappa > 0$ such that, using the notation defined below, the function $g_\kappa : P^\varepsilon \setminus (P^\circ \setminus [-\kappa, \kappa] \times \mathbb{R}) \rightarrow \mathbb{R}$ defined as $g_\kappa(z_i^\pm) = f(z_i^\pm)$ for $i = 0, n$, $g_\kappa(z_i^\pm) = f(z_i)$ for $i = 1, \dots, n-1$ and

$$g_\kappa(x) = \begin{cases} f(x), & \text{if } x \in P^\varepsilon \setminus P, \\ \alpha g(z_i^+) + \beta g(z_i^-) + \gamma g(z_{i+1}^+), & \text{for } x = \alpha z_i^+ + \beta z_i^- + \gamma z_{i+1}^+, \\ \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = 1, \\ \alpha g(z_i^-) + \beta g(z_{i+1}^-) + \gamma g(z_{i+1}^+), & \text{for } x = \alpha z_i^- + \beta z_{i+1}^- + \gamma z_{i+1}^+, \\ \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = 1 \end{cases}$$

is $(L + \delta)$ -Lipschitz and such that $|f - g_\kappa| < \delta$ on \mathbb{R}^2 .

Here we denoted $z_0^\pm := (\pm\kappa, \frac{a+c}{2} \pm \frac{\kappa(a-c)}{2})$, $z_n^\pm := (\pm\kappa, \frac{b+d}{2} \pm \frac{\kappa(b-d)}{2})$, $z_i^\pm := (\pm\kappa, a_i)$ for $i = 1, \dots, n-1$ and $z_i := (0, a_i)$ for $i = 0, \dots, n$.

Proof. Without any loss of generality we can suppose $L = 1$. Denote P_i^ε the connectivity component of $P^\varepsilon \setminus P^\circ$ containing z_i , $i = 0, n$. When we will have a_i found we will denote $P_i = \text{co}\{c_i^\pm, c_{i+1}^\pm\}$ for $i = 0, \dots, n-1$.

First use lemma 3.4 to find $a_1 \in B(a_0, \frac{\min(|a_0 - a_n|, 1)}{2})$ and $a_{n-1} \in B(a_n, \frac{\min(|a_0 - a_n|, 1)}{2})$ and $\kappa_1 > 0$ such that for every $\kappa > 0$ the function $g|_{P_0^\varepsilon \cup P_0}$ and $g|_{P_n^\varepsilon \cup P_{n-1}}$ are both $(1 + \delta)$ -Lipschitz and such that $|f - g_\kappa| < \delta$ on $P^\varepsilon \cup P_0 \cup P_{n-1}$.

Observe that for every $u_0 \in P_0^\varepsilon \cup P_0$ and every $u_n \in P_n^\varepsilon \cup P_{n-1}$ we have

$$\begin{aligned} \frac{|g_\kappa(u_0) - g_\kappa(u_n)|}{|u_0 - u_n|} &\leq \frac{|g_\kappa(u_0) - g_\kappa(z_0)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_0) - g_\kappa(z_n)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_n) - g_\kappa(u_n)|}{|u_0 - u_n|} \\ &\leq \frac{|u_0 - z_0|}{|u_0 - u_n|} + \frac{|z_0 - z_n|}{|u_0 - u_n|} + \frac{|z_n - u_n|}{|u_0 - u_n|}. \end{aligned}$$

and since the last formula can be smaller than $1 + \delta$ when we assume $|a_0 - a_1|$ and $|a_{n-1} - a_n|$ to be small enough, we can additionally assume that $g|_{P^\varepsilon \cup P_0 \cup P_{n-1}}$ is $(1 + \delta)$ -Lipschitz.

Next, note that the function $g_\kappa|_{[z_1, z_{n-1}]}$ is actually independent on κ and that it is 1-Lipschitz for any choice of a_2, \dots, a_{n-2} (this is because in one dimension the affine extension never increases the Lipschitz constant). This also means that for $S = \text{co}\{c_1^\pm, c_{n-1}^\pm\}$ we have $g_\kappa|_S$ is 1-Lipschitz for any choice of a_2, \dots, a_{n-2} as well. Put $\alpha = \text{dist}(S, P^\varepsilon \setminus P)$, we can assume κ_2 to be small enough that $1 > \alpha > 0$ (here we used the fact that $|a_0 - a_1|, |a_{n-1} - a_n| \leq \frac{1}{2}$). Consider n big enough such that $\frac{|a_1 - a_{n-1}|}{n-1} \leq \frac{\alpha\delta}{4}$, put $a_i = a_1 + \frac{i|a_1 - a_{n-1}|}{n-1}$ and pick $\kappa_3 < \min(\kappa_2, \frac{\alpha\delta}{4})$. Then for $\kappa < \kappa_3$ and $a \in S$

$$\begin{aligned} |g_\kappa(a) - f(a)| &\leq |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - f(z_i)| + |f(z_i) - f(a)| \\ (3.2) \quad &\leq |a - z_i| + 0 + |a - z_i| \leq \frac{\delta}{2} < \delta, \end{aligned}$$

where i is chosen such that $a \in P_i$.

To finish the proof we need to observe that for $\kappa < \kappa_3$ the function g_κ is $(1 + \delta)$ -Lipschitz. Since $S \cup P_0 \cup P_{n-1}$ is convex, the remaining case we have to consider is $a \in S$ and $b \in P^\varepsilon \setminus P$. Find i such that $a \in P_i$. With this choice we have $|a - z_i| \leq \frac{\alpha\delta}{2}$ and therefore

$$|b - z_i| \leq |a - b| + |a - z_i| \leq |a - b| + \frac{\alpha\delta}{2} \leq (1 + \delta) |a - b|.$$

Now,

$$\begin{aligned}
|g_\kappa(a) - g_\kappa(b)| &\leq |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - g_\kappa(b)| \\
&\leq \frac{\delta\alpha}{2} + |f(z_i) - f(b)| \leq \frac{\delta}{2}|a - b| + |b - z_i| \\
&\leq \frac{\delta}{2}|a - b| + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| \leq (1 + \delta)|a - b|.
\end{aligned}$$

□

Lemma 3.6. *Let $1 > \varepsilon > 0$ and $\alpha, L > 0$. Let f be a L -Lipschitz function on $[-1, 1]^2$ which is affine on both $[-1, 0] \times [-1, 1]$ and $[0, 1] \times [-1, 1]$ (and equal to affine functions f_1 and f_2 , respectively). Put*

$$A_1 = [-1, 0] \times [-1, -1/2], A_2 = [0, 1] \times [1/2, 1],$$

$$B_1^\varepsilon = [0, \varepsilon] \times [-1, \varepsilon], B_2^\varepsilon = [-\varepsilon, 0] \times [-\varepsilon, 1]$$

and

$$A = A_1 \cup A_2 \cup B_1^\varepsilon \cup B_2^\varepsilon.$$

Then either f is convex on $[-1, 1]^2$ or the function $g_\varepsilon : A \rightarrow \mathbb{R}$ defined as

$$g(x) = \begin{cases} f_1(x), & \text{if } x \in A_1 \cup B_1^\varepsilon, \\ f_2(x), & \text{if } x \in A_2 \cup B_2^\varepsilon. \end{cases}$$

is locally convex on A . Moreover, if ε is small enough, g_ε is $(L + \alpha)$ -Lipschitz and $|g_\varepsilon - f| < \alpha$ on A .

Proof. Direct computation. □

Lemma 3.7. *Let $L, \alpha > 0$ and $1 > \gamma > \varepsilon > 0$. Let f be a L -Lipschitz function on $[-4, 4]^2 \cup [4, 5] \times [1, 2]$ which is affine on both $[-4, 0] \times [-4, 4]$ and $[0, 4] \times [-4, 4] \cup [4, 5] \times [1, 2]$ (and equal to affine functions f_1 and f_2 , respectively). Put*

$$A_1 = [0, \gamma] \times [-3, -2], A_2 = [\gamma, \gamma + \varepsilon] \times [-3, 0], A_3 = [\gamma - \varepsilon, \gamma] \times [-1, 2],$$

$$A_4 = [\gamma, 4] \times [1, 2], B_1 = [-4, 0] \times [-4, 4], B_2 = [4, 5] \times [1, 2],$$

and

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2.$$

Then either f is locally convex on $[-4, 4]^2 \cup [4, 5] \times [1, 2]$ or the function

$$g(x) = \begin{cases} f_1(x), & \text{if } x \in A_1 \cup A_2 \cup B_1, \\ f_2(x) + \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4}(x \cdot (1, 0) - 4), & \text{if } x \in A_3 \cup A_4, \\ f_2(x), & \text{if } x \in B_2, \end{cases}$$

is $(L + \alpha)$ -Lipschitz, locally convex on A and $|f - g| < \alpha$ on A , if ε and γ are small enough.

Proof. Without any loss of generality we can suppose $L = 1$. First we prove that g is continuous on A . To do this we need to prove that

$$(3.3) \quad f_1(\gamma, a) = f_2(\gamma, a) + \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4}((\gamma, a) \cdot (1, 0) - 4)$$

whenever $(\gamma, a) \in A$ and that

$$(3.4) \quad f_2(4, a) = f_2(4, a) + \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4}((4, a) \cdot (1, 0) - 4)$$

whenever $(4, a) \in A$. Define an affine function f_3 on \mathbb{R}^2 as

$$f_3(u, v) = \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4}((u, v) \cdot (1, 0) - 4).$$

To prove (3.3) we can write

$$\begin{aligned}
g(\gamma, a) &= f_2(\gamma, a) + f_3(\gamma, a) \\
&= f_2(\gamma, a) + \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4} \cdot (\gamma - 4) \\
&= f_2(\gamma, a) + f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_2(0, 0) \\
&= f_2(\gamma, a) + f_1(\gamma, a) - f_1(0, a) - f_2(\gamma, a) + f_2(0, a) \\
&= f_2(\gamma, a) + f_1(\gamma, a) - f_1(0, a) - f_2(\gamma, a) + f_1(0, a) = f_1(\gamma, a).
\end{aligned}$$

To prove (3.4) we can write

$$\begin{aligned}
g(4, a) &= f_2(4, a) + f_3(4, a) \\
&= f_2(4, a) + \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_1(0, 0)}{\gamma - 4} (4 - 4) = f_2(4, a).
\end{aligned}$$

Next note that since both f_1 and f_2 are 1-Lipschitz we have

$$(3.5) \quad g \text{ is 1-Lipschitz on } B_1 \cup A_1 \cup A_2,$$

and

$$(3.6) \quad g \text{ is 1-Lipschitz on } B_2,$$

also since additionally f_3 is constant on all lines parallel to y -axis and since

$$\frac{f_3(\gamma, 0) - f_3(4, 0)}{4 - \gamma} \leq \frac{f_1(\gamma, 0) - f_1(0, 0) - f_2(\gamma, 0) + f_2(0, 0) - 0}{3} \leq \frac{2\gamma}{3} \leq \gamma.$$

we have

$$(3.7) \quad g \text{ is } (1 + \gamma)\text{-Lipschitz on } A_4 \cup A_3.$$

and

$$(3.8) \quad |g - f_2| \leq 4\gamma \text{ on } A_4 \cup A_3.$$

Now, if $x \in B_1$ and $y \in A_3$ then $g(x) = f_1(x)$, $|g(y) - f_1(y)| \leq 3\varepsilon$ and $|x - y| \geq \gamma - \varepsilon$ and therefore

$$|g(x) - g(y)| \leq |g(x) - f_1(y)| + |f_1(y) - g(y)| \leq |x - y| + 3\varepsilon \leq \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}.$$

So

$$(3.9) \quad g \text{ is } \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}\text{-Lipschitz on } B_1 \cup A_3.$$

If $x \in B_1$ and $y \in A_4$ then $g(x) = f_1(x)$, $f(y) \leq g(y) \leq f_1(y)$ and therefore

$$(3.10) \quad g \text{ is 1-Lipschitz on } B_1 \cup A_4.$$

Using (3.6) and (3.7) and continuity of g we obtain that

$$(3.11) \quad g \text{ is } (1 + \gamma)\text{-Lipschitz on } A_2 \cup A_3 \text{ and on } B_2 \cup A_4.$$

Finally, if $x \in A_1 \cup A_2$ and $y \in A_4 \cup B_2$ or $x \in A_1$ and $y \in A_3 \cup A_4 \cup B_2$ we have

$$(3.12) \quad |g(x) - f_2(x)| \leq 2(\gamma + \varepsilon) \leq 4\gamma, \quad |g(y) - f_2(y)| \leq 4\gamma$$

and $|x - y| \geq 1$. This implies

$$\begin{aligned}
(3.13) \quad |g(x) - g(y)| &\leq |g(x) - f_2(x)| + |f_2(x) - f_2(y)| + |f_2(y) - g(y)| \\
&\leq 4\gamma + |x - y| + 4\gamma \leq (1 + 8\gamma)|x - y|.
\end{aligned}$$

Now, according to (3.5), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) it is sufficient to choose $\frac{\alpha}{4} > \gamma > \varepsilon > 0$ small enough such that

$$\max \left(1 + 8\gamma, \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon} \right) < 1 + \alpha$$

to obtain that g is $(1 + \alpha)$ -Lipschitz on A and $|f - g| < \alpha$ on A . \square

Lemma 3.8. *Under the assumptions of Lemma 3.5 there is a $\frac{1}{2} > \kappa > 0$, $R \subset P^\circ \cap \mathbb{R} \times (-\kappa, \kappa)$ and a function $h : (P^\varepsilon \setminus P) \cup R \rightarrow \mathbb{R}$ such that:*

- (a) $R \in \mathcal{Q}$,
- (b) $h = f$ on $P^\varepsilon \setminus P^\circ$,
- (c) h is locally convex on $(P^\varepsilon \setminus P^\circ) \cup R$,
- (d) $(P^\varepsilon \setminus P) \cup R$ is connected,
- (e) h is piecewise affine on $(P^\varepsilon \setminus P^\circ) \cup R$,
- (f) h is $(L + \delta)$ -Lipschitz.

Proof. Without any loss of generality we can suppose $L = 1$. Let κ, z_i, g_κ as in Lemma 3.5, but with $\frac{\delta}{2}$ in the place of δ . Consider the sets

$$X = [-4, 4]^2 \cup [4, 5] \times [1, 2] \quad \text{and} \quad Y = [-1, 1]^2.$$

Find similarities $\Psi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $i = 0, \dots, n$ such that if we put $M_i = \Psi_i(X)$, $i = 0, n$ and $M_i = \Psi_i(Y)$, $i = 1, \dots, n-1$ we have

- (A) $M_i \cap M_j = \emptyset$ if $i \neq j$,
- (B) $\Psi_0([-4, 0] \times [-4, 4]) \subset P^\varepsilon \setminus P^\circ$,
- (C) $\Psi_n([-4, 0] \times [-4, 4]) \subset P^\varepsilon \setminus P^\circ$,
- (D) $M_i \subset \mathbb{R} \times (-\kappa, \kappa)$,
- (E) $[z_i^-, z_i^+] \subset \Psi_i(\{0\} \times \mathbb{R})$,
- (F) Ψ_i preserves orientation for $i = 1, \dots, n-1$

Put $\Omega = \min_{i \neq j} \text{dist}(M_i, M_j)$, note that $\Omega > 0$ due to property (A). Define

$$T_i := \text{co}\{\Psi_i((1, \frac{1}{2}), \Psi_i(1, 1), \Psi_{i+1}((-1, -\frac{1}{2}), \Psi_{i+1}(-1, -1))\},$$

for $i = 1, \dots, n-2$,

$$T_0 := \text{co}\{\Psi_0(5, 1), \Psi_0(5, 2), \Psi_1(-1, -\frac{1}{2}), \Psi_1(-1, -1)\}$$

and

$$T_{n-1} := \text{co}\{\Psi_n(5, 1), \Psi_n(5, 2), \Psi_{n-1}(1, \frac{1}{2}), \Psi_{n-1}(1, 1)\}.$$

and put

$$(3.14) \quad R := \left(\bigcup_{i=0}^{n-1} T_i \right) \cup \left(\bigcup_{i=0}^n M_i \right).$$

Let ρ_i be scaling ratio of Ψ_i . Let g_i , $i = 1, \dots, n-1$ be the function g from Lemma 3.6 with $\alpha = \frac{\Omega \delta \rho_i}{4}$ (and corresponding ε) and with $f_1(x) = \rho_i \kappa \circ \Psi_i$ and $f_2(x) = \rho_i \kappa \circ \Psi_i$ (with the exception if g_κ is already convex on M_i , in which case we put $g_i = g_\kappa|_{M_i}$), let g_0 be the function g from Lemma 3.7 with $\gamma = \frac{\Omega \delta \rho_i}{4}$ (and corresponding ε and γ) and with $f_1 = \rho_0 \kappa \circ \Psi_0$ and $f_2 = \rho_0 \kappa \circ \Psi_0$ and finally, let g_n be the function g from Lemma 3.7 with $\gamma = \frac{\Omega \delta \rho_i}{4}$ (and corresponding ε and γ) and with $f_1 = \rho_n \kappa \circ \Psi_n$ and $f_2 = \rho_n \kappa \circ \Psi_n$.

Consider now the function h defined by the formula

$$h = \begin{cases} \frac{1}{\rho_i} g_i \circ \Psi_i^{-1} & \text{on } M_i \\ g_\kappa & \text{otherwise.} \end{cases}$$

Property (a) follows from (3.14) and the fact that every M_i and every T_i is a polygon. Properties (b), (c) and (e) follow directly from the construction and corresponding properties of the functions g_i and property (d) is obvious. We will now finish the proof by proving property (f).

So suppose that $a, b \in (P^\varepsilon \setminus P) \cup R$. we need to prove that $|h(a) - h(b)| \leq (1 + \delta)|a - b|$. We can additionally suppose that either a or b belongs to some M_i since otherwise there is nothing to prove. We will prove only the case $a \in M_i$, $b \in M_j$, $i \neq j$, the other cases can be proved following the same lines. By Lemma 3.6 (for $i = 1, \dots, n - 1$) and Lemma 3.7 (for $i = 0, n$) we can now write

$$\begin{aligned} |h(a) - h(b)| &\leq |h(a) - g_\kappa(a)| + |g_\kappa(a) - g_\kappa(b)| + |g_\kappa(b) - h(b)| \\ &< \frac{1}{\rho_i} \cdot \frac{\Omega\delta\rho_i}{4} + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| + \frac{1}{\rho_j} \cdot \frac{\Omega\delta\rho_j}{4} \\ &\leq \frac{\delta}{2}|a - b| + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| = (1 + \delta)|a - b|, \end{aligned}$$

which is what we need. \square

Proof of Lemma 3.2. Without any loss of generality we can suppose $L = 1$. Let V be the set of all points $v \in \partial P$ with the property that there is some $\varepsilon_v > 0$ such that $P \cap B(v, \varepsilon_v)$ is similar to $\{(x, y) : x \geq 0\} \cap B(0, 1)$ and that f is affine on $P \cap B(v, \varepsilon_v)$. Since $P \in \mathcal{Q}$, the set $\partial P \setminus V$ is finite and we can without any loss of generality assume that $l(\varepsilon) \cap (\partial P \setminus V) = \emptyset$.

This means that the closure of every bounded component C_i of $P \cap l(\varepsilon)$ is a similar copy of

$$\text{co}\{(-1, a_i), (-1, b_i), (1, c_i), (1, d_i)\} =: P_i$$

for some $a_i < b_i$, $c_i < d_i$ and such that for some $\varepsilon_i > 0$ f is locally affine on $P_i^{\varepsilon_i} \setminus P$, where

$$P_i^{\varepsilon_i} := \text{co}\{(-1, a_i - \varepsilon_i), (-1, b_i + \varepsilon_i), (1, c_i - \varepsilon_i), (1, d_i + \varepsilon_i)\}.$$

Then

$$\alpha = \min_{i \neq j} \text{dist}(C_i, C_j) > 0$$

Let Ψ_i be a similarity between C_i and S_i and let κ_i , R_i and h_i be κ , R and h as obtained from Lemma 3.8 for $\varepsilon = \varepsilon_i$, $P = P_i$, $f = \rho_i g \circ \Psi_i$ and $\delta = \frac{\min(\alpha, \varepsilon_i, 1)\rho_i\varepsilon}{4}$, where ρ_i is the similarity ratio on Ψ_i .

Put $Q = P \setminus (\bigcup R_i)$ and define $\tilde{h} : Q^c \rightarrow \mathbb{R}$ by

$$\tilde{h} = \begin{cases} \frac{1}{\rho_i} h_i \circ \Psi_i^{-1} & \text{on } R_i \\ g & \text{otherwise.} \end{cases}$$

Let K be the Lipschitz constant of \tilde{h} , the using the Kirszbraun theorem on extensions of Lipschitz functions we can find a K -Lipschitz function h on \mathbb{R}^2 such that $h = \tilde{h}$ on P^c .

Now, property (1) follows directly from the definition of Q and (a) in Lemma 3.8, property (2) from the definition of h and (b) in Lemma 3.8 and property (3) from (d) in Lemma 3.8.

It remains to prove that the pair (Q, h) is $(1 + \varepsilon)$ -good. The local convexity and piecewise affinity of h on Q^c follows from (c) and (e) in Lemma 3.8 and the corresponding properties of g , so the proof will be finished, if we verify that $K \leq (1 + \varepsilon)$.

To do this pick $a, b \in \mathbb{R}^2$, we need to prove that $|h(a) - h(b)| \leq (1 + \varepsilon)|a - b|$.

We can additionally suppose that either a or b belongs to some R_i since otherwise there is nothing to prove. We will prove only the case $a \in R_i$, $b \in R_j$, $i \neq j$, the other cases can be proved following the same lines.

Using the definition of h , namely property (f) from Lemma 3.8 we can now write

$$\begin{aligned} |h(a) - h(b)| &= |h_i(a) - h_j(b)| \leq |h_i(a) - f(a)| + |f(a) - f(b)| + |f(b) - h_j(b)| \\ &\leq \frac{1}{\rho_i} \cdot \frac{\min(\alpha, \varepsilon_i) \rho_i \varepsilon}{4} + \left(1 + \frac{\varepsilon}{4}\right) \cdot |a - b| + \frac{1}{\rho_j} \cdot \frac{\min(\alpha, \varepsilon_j) \rho_j \varepsilon}{4} \\ &\leq \frac{2\varepsilon}{4} |a - b| + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| < (1 + \delta) |a - b|. \end{aligned}$$

□

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